THE SADDLE-POINT METHOD IN $\mathbb{C}^N$

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Abstract: For any positive integer N, the complex Morse lemma yields an asymptotic series expansion, as the real parameter $\tau \to +\infty$, of an integral

$$I(\tau) = \int_\Gamma e^{\tau\, h(z_1,\ldots,z_N)} g(z_1,\ldots,z_N)\,\mathrm{d}z_1 \cdots \mathrm{d}z_N,$$

where Γ is a manifold in $\mathbb{C}^N$ and where g and h are holomorphic functions in an open subset of $\mathbb{C}^N$ containing Γ, provided Γ contains a nondegenerate saddle-point $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ of $e^{h(z_1,\ldots,z_N)}$ at which $\mathrm{Re}\, h(z_1,\ldots,z_N)$ is maximal. Asymptotic formulae for integrals $I(\tau)$ as above have several applications to complex analysis and number theory. In general, however, the given integration manifold Γ fails to contain a saddle-point of $e^{h(z_1,\ldots,z_N)}$. Thus the main difficulty to apply the saddle-point method to the asymptotic study of $I(\tau)$ is to prove that Γ can be deformed to a new integration manifold Λ, equivalent to Γ by Poincaré's theorem and hence preserving the value of the integral, and containing the relevant saddle-point $\left(z_1^{(0)},\ldots,z_N^{(0)}\right)$ as required. In my talk I will explain under which assumptions the deformation of Γ into Λ can be ensured, and I will sketch out some applications of asymptotic expansions of $I(\tau)$ to Diophantine approximation.