ASYMPTOTICS TO ALL ORDERS OF THE HURWITZ ZETA FUNCTION

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ABSTRACT: The approximate functional equation for the Riemann zeta function $\zeta(s)$ states that [12]:

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \chi(s) \sum_{n\leq y} \frac{1}{n^{1-s}} + O\Big(x^{-\sigma} + |t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\Big), \tag{1}$$

where Γ denotes the gamma function of a complex variable s, the entire function χ is defined by $\chi(s) := \frac{(2\pi)^s}{\pi}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$, and the variables $x, y, s = \sigma + it$ satisfy

$$xy = \frac{t}{2\pi}, \quad 0 < \sigma < 1, \quad t \to \infty. \tag{2}$$

The result (1) can be interpreted as a first-order asymptotic expansion of the Riemann zeta function: it gives us an expression for $\zeta(s)$ in the form of a finite closed-form series followed by a big-O term bounding the remainder. A natural question then is to ask whether this expansion can be extended to *all* orders. The behaviour of $\zeta(\sigma + it)$ for large values of t is a very important problem, relating e.g. to the Lindelöf Hypothesis, so a large-t asymptotic expansion could be very valuable.

In the important special case of $x = y = \sqrt{t/2\pi}$, large-t asymptotics to all orders for the Riemann zeta function $\zeta(s)$, i.e. for the error term in (1), were proven in the classical paper of Siegel [11], following Riemann's unpublished notes. The recent paper [6] proves a more general result for large-t asymptotics to all orders, valid for any x, y satisfying (2). The starting point of this study was the following exact integral formula, proved in Theorem 2.1 of [6]:

$$\begin{aligned}\zeta(s) = \chi(s)\Bigg[\sum_{n\leq \eta/2\pi} n^{s-1} + \frac{1}{(2\pi)^s}\Bigg(-\frac{\eta^s}{s} + e^{i\pi s/2}\int_{-i\eta}^{\infty e^{i\phi_1}} \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z \\ + e^{-i\pi s/2}\int_{i\eta}^{\infty e^{i\phi_2}} \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z\Bigg)\Bigg].\end{aligned} \tag{3}$$

Note that this formula is written in terms of a new parameter η and is valid for

$$0 < \eta < \infty, \qquad -\tfrac{\pi}{2} < \phi_1, \qquad \phi_2 < \tfrac{\pi}{2}, \qquad s \in \mathbb{C}.$$

In the present work we consider the *Hurwitz zeta function*, a two-variable generalisation of the Riemann zeta function defined by

$$\zeta(x,s) := \sum_{n=0}^{\infty}(n+x)^{-s}, \qquad \mathrm{Re}(x)>0,\ s=\sigma+it,\ \sigma>1,\ t\in\mathbb{R},$$

and its slightly modified form which we will call the *modified Hurwitz zeta function*, defined by

$$\zeta_1(x,s) := \sum_{n=1}^{\infty}(n+x)^{-s}, \qquad \mathrm{Re}(x)>-1,\ s=\sigma+it,\ \sigma>1,\ t\in\mathbb{R}.$$

Both of these can be extended by analytic continuation to meromorphic functions on the whole complex plane, and they are clearly related by the identity

$$\zeta(x,s) = \frac{1}{x^s} + \zeta_1(x,s), \qquad \mathrm{Re}(x)>0,\ s\in\mathbb{C}.$$

They are also generalisations of the standard Riemann zeta function: it is clear that

$$\zeta(s) = \zeta_1(0,s), \qquad s\in\mathbb{C}.$$

But the existence of the additional parameter x occurring in the Hurwitz zeta function gives rise to interesting results which do not have analogues for $\zeta(s)$; see for example [13], [2], [1], [7], [8], and p. 73 in [3]. Thus it is useful to consider the extension of the results of [6] to prove asymptotics to all orders of the Hurwitz zeta function.

An analogous formula to the approximate functional equation (1) for the modified Hurwitz zeta function has been proved e.g. in [10]. But the argument of Siegel [11] does not extend easily to the Hurwitz function, as it requires properties specific to the Riemann zeta function. In this work, based on our paper [4], we present analogous results with those of [6] for the function $\zeta_1(x,s)$, obtaining large-t asymptotics to all orders for the modified Hurwitz zeta function. Our starting point is the following exact formula, analogous to (3):

$$\zeta_1(x,s) = \chi(s)\Bigg(\sum_{m\le\eta/2\pi} e^{-2\pi i m x} m^{s-1} - \frac{e^{-i\pi s/2}}{(2\pi)^s}\int_{\hat{C}_\eta^0} \frac{e^{(1+x)z}-e^{-xz}}{1-e^z} z^{s-1}\,\mathrm{d}z + \frac{e^{i\pi s/2}}{(2\pi)^s}\int_{-i\eta}^{\infty e^{i\phi_2}} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z + \frac{e^{-i\pi s/2}}{(2\pi)^s}\int_{i\eta}^{\infty e^{i\phi_1}} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z\Bigg), \tag{4}$$

valid for

$$0<\eta<\infty, \quad -\tfrac{\pi}{2}<\phi_1,\phi_2<\tfrac{\pi}{2}, \quad 0<\sigma\le 1, \quad 0<t<\infty, \quad 0<x<\infty.$$

We analyse this using an integration by parts method, as seen in [9]. Thus we are using arguments from applied analysis in order to derive results in number theory. This cross-disciplinary approach is one which we hope will yield many interesting new results.

The problem we are considering is more advanced than that considered in [6] for the Riemann zeta function; the first of the three integral terms in (4) does not appear at all in (3), being easy to evaluate explicitly in the case $x = 0$, and this turns out to be the most difficult one to analyse.

However, surprisingly, the analysis and methods we use turn out to be simpler than those employed in [6], in the following ways.

1. Starting from (3), the three cases $\eta \ll t$, $\eta \sim t$, $\eta \gg t$ were analysed separately in [6], using different methods in each case. Our approach is more unified, using the same basic method for all three cases. We require a certain condition to be placed on η, but this condition is not very restrictive and we hope to be able to use the ideas of [5] to eliminate it entirely.

2. Some of the integrals in [6] contained stationary points which had to be analysed using different techniques. We have rewritten each such integral as the combinations of one integral which can be computed explicitly and one which does not contain stationary points, thus avoiding this complication.

3. In the $\eta \ll t$ case, the method used in [6] involved a complex Siegel-type contour. Our unified approach uses only integration by parts, with various tricks to make the calculations slicker.

In the present work, we shall show how to derive the equation (4), as well as outlining the asymptotic analysis to all orders of each of the three integrals in the RHS of (4), which can then be put together to yield the final result for $\zeta_1(x, s)$.

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